

Concept of Lie Derivative of Spinor Fields. A Geometric Motivated Approach

Waldyr A. Rodrigues Jr., Rafael F. Leão and Samuel A. Wainer
IMECC-UNICAMP

walrod@ime.unicamp.br leao@ime.unicamp.br samuelwainer@ime.unicamp.br

November 30 2015

Contents

1	Introduction	2
2	Preliminaries	2
3	On the Concept of Lie Derivatives of Spinor Fields in Lorentzian Manifolds	7
4	The Spinor Lie Derivative $\overset{s}{\mathcal{L}}_{\xi}$	12
4.1	Spinor Images of Clifford and Spinor Fields	12
4.2	Spinor Derivative of Clifford and Spinor Fields	12
5	The Spinor Lie Derivative Written in Terms of Covariant Derivatives	15
5.1	Calculation of $\overset{s}{\mathcal{L}}_{\xi}\psi_{\Xi_0}$ in Coordinates	15
5.2	Spinor Lie Derivative of Covariant Dirac Spinor Fields	16
6	The Spinor Lie derivative of the Metric Field	16
7	Other Definitions of Lie Derivatives of Spinor Fields.	17
8	Conclusions	19

Abstract

In this paper using the Clifford bundle $(\mathcal{Cl}(M, \mathbf{g}))$ and spin-Clifford bundle $(\mathcal{Cl}_{\text{Spin}_{1,3}^e}(M, \mathbf{g}))$ formalism, which permit to give a meaningful representative of a Dirac-Hestenes spinor field (even section of $\mathcal{Cl}_{\text{Spin}_{1,3}^e}(M, \mathbf{g})$) in the Clifford bundle, we give a geometrical motivated definition for the Lie derivative of spinor fields in a Lorentzian structure (M, \mathbf{g}) where M is a manifold such that $\dim M = 4$, \mathbf{g} is Lorentzian of signature $(1, 3)$. Our

Lie derivative, called the spinor Lie derivative (and denoted $\overset{s}{\mathcal{L}}_{\xi}$) is given by nice formulas when applied to Clifford and spinor fields, and moreover $\overset{s}{\mathcal{L}}_{\xi}g = 0$ for any vector field ξ . We compare our definitions and results with the many others appearing in literature on the subject.

1 Introduction

Our principal aim in this paper is, using the Clifford and spin-Clifford bundles formalism, to give a geometrical motivated definition for the Lie derivative of spinor fields in a Lorentzian structure (M, g) , that will be defined below.

In Section 2 we recall some key definitions and some propositions that will suggest us how to define the spinor image of Clifford and spinor fields using the spinor lifting of an integral curve of a vector field, as set in Definition (16). In section 3 we expose the main problem concerning the definition of an appropriate definition for the Lie derivative of spinor fields. We then present a proposition that permit us to calculate the usual Lie derivative (\mathcal{L}_{ξ}) of a cotrad basis in the direction of an arbitrary vector field ξ in two different ways, the usual way, and one of them making use of the concept of the spinor lifting (Definition 16) of an integral curve of ξ in $P_{Spin_{1,3}}(M, g)$. It is this way of obtaining the Lie derivative of a cotrad basis that suggested us to give in Section 4.1 a geometrical motivated definition of spinor images of Clifford and spinor fields introducing the spinor mapping ${}^s h_t$ and next to define in Section 4.2 the *spinor Lie derivative* (denoted $\overset{s}{\mathcal{L}}_{\xi}$) of Clifford and spinor fields and then to calculate the explicit simple and nice formulas for those objects. The idea of Section 5 is to write the spinor Lie derivative in terms of covariant derivatives in such way that we can relate our construction with the literature, as it appears in [22]. In particular we evaluate in Section 5.1 the spinor Lie derivative of a representative of a DHSF in local coordinates and in Section 5.2 we write the spinor Lie derivative for covariant Dirac spinor fields. In Section 6 we show that $\overset{s}{\mathcal{L}}_{\xi}g = 0$, for any arbitrary vector field ξ . A definition of Lie derivative that annihilates g has been given firstly by Bourguignon & Gauduchon [3], but our approach is very different from the one used by those authors. The main proposal of Section 7 is to comment on some different approaches to the Lie derivative of spinor fields, with conflicting views appearing in the literature, and how our geometrical approach intersects these and present our future prospects. Finally, in Section 8 we present our conclusions.

2 Preliminaries

Here, M refers¹ to a four dimensional, real, connected, paracompact and non-compact manifold. We define a Lorentzian manifold as a pair (M, g) , where $g \in \sec T_2^0 M$ is a Lorentzian metric of signature $(1, 3)$, i.e., $\forall x \in M, T_x M \simeq$

¹Unless, explicitly stated.

$T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space. We define a Lorentzian spacetime M as pentuple $(M, \mathbf{g}, \mathbf{D}, \tau_{\mathbf{g}}, \uparrow)$, where $(M, \mathbf{g}, \tau_{\mathbf{g}}, \uparrow)$ is an oriented Lorentzian manifold (oriented by $\tau_{\mathbf{g}}$) and time oriented by \uparrow , and \mathbf{D} is the Levi-Civita connection of \mathbf{g} . Let $\mathcal{U} \subseteq M$ be an open set covered by coordinates $\{x^\mu\}$. Let $\{e_\mu = \partial_\mu\}$ be a coordinate basis of $T\mathcal{U}$ and $\{\vartheta^\mu = dx^\mu\}$ the dual basis on $T^*\mathcal{U}$, i.e., $\vartheta^\mu(\partial_\nu) = \delta_\nu^\mu$. If $\mathbf{g} = g_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$ is the metric on $T\mathcal{U}$ we denote by $\mathbf{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$ the metric of $T^*\mathcal{U}$, such that $g^{\mu\rho}g_{\rho\nu} = \delta_\nu^\mu$. We introduce also $\{\boldsymbol{\vartheta}^\mu\}$ and $\{\boldsymbol{\vartheta}_\mu\}$, respectively, as the reciprocal bases of $\{e_\mu\}$ and $\{\vartheta_\mu\}$, i.e., we have

$$\mathbf{g}(\partial_\nu, \partial^\mu) = \delta_\nu^\mu, \quad \mathbf{g}(\vartheta^\mu, \vartheta_\nu) = \delta_\nu^\mu. \quad (1)$$

In what follows $\mathbf{P}_{\text{SO}_{1,3}^e}(M, \mathbf{g})$ ($P_{\text{SO}_{1,3}^e}(M, \mathbf{g})$) denotes the principal bundle of oriented Lorentz tetrads (cotetrads).

Definition 1 A spin structure for a general m -dimensional manifold M consists of a principal fiber bundle $\pi_s : P_{\text{Spin}_{p,q}^e}(M, \mathbf{g}) \rightarrow M$, (called the Spin Frame Bundle) with group $\text{Spin}_{p,q}^e$ and a map

$$\Lambda : P_{\text{Spin}_{p,q}^e}(M, \mathbf{g}) \rightarrow P_{\text{SO}_{p,q}^e}(M, \mathbf{g}), \quad (2)$$

satisfying the following conditions:

- (i) $\pi(\Lambda(p)) = \pi_s(p), \forall p \in P_{\text{Spin}_{p,q}^e}(M, \mathbf{g})$, where π is the projection map of the bundle $\pi : P_{\text{SO}_{p,q}^e}(M, \mathbf{g}) \rightarrow M$.
- (ii) $\Lambda(pu) = \Lambda(p)\text{Ad}_u, \forall p \in P_{\text{Spin}_{p,q}^e}(M, \mathbf{g})$ and $\text{Ad} : \text{Spin}_{p,q}^e \rightarrow \text{SO}_{p,q}^e$, $\text{Ad}_u(a) = uau^{-1}$.

Definition 2 Any section of $P_{\text{Spin}_{p,q}^e}(M, \mathbf{g})$ is called a spin frame field (or simply a spin frame). We shall use the symbol $\Xi \in \text{sec } P_{\text{Spin}_{p,q}^e}(M, \mathbf{g})$ to denote a spin frame.

In this work we will assume that exists a spin structure on the 4-dimensional Lorentzian manifold (M, \mathbf{g}) , what implies that M is parallelizable, i.e., $P_{\text{SO}_{1,3}^e}(M, \mathbf{g})$ is trivial, because of the following result:

Theorem 3 For a 4-dimensional Lorentzian manifold (M, \mathbf{g}) , a spin structure exists if and only if $P_{\text{SO}_{1,3}^e}(M, \mathbf{g})$ is a trivial bundle.

Proof. See Geroch [10] ■

The Clifford bundle of differential forms of a Lorentzian manifold (M, \mathbf{g}) is the bundle of algebras $\mathcal{Cl}(M, \mathbf{g}) = \bigsqcup_{x \in M} \mathcal{Cl}(T_x^*M, \mathbf{g}_x)$.

We know that² [29]:

$$\mathcal{Cl}(M, \mathbf{g}) = P_{\text{SO}_{1,3}^e}(M, \mathbf{g}) \times_\rho \mathbb{R}_{1,3} = P_{\text{Spin}_{1,3}^e}(M, \mathbf{g}) \times_{\text{Ad}} \mathbb{R}_{1,3}, \quad (3)$$

²Where $\text{Ad} : \text{Spin}_{1,3}^e \rightarrow \text{End}(\mathbb{R}_{1,3})$ is such that $\text{Ad}(u)a = uau^{-1}$. And $\rho : \text{SO}_{1,3}^e \rightarrow \text{End}(\mathbb{R}_{1,3})$ is the natural action of $\text{SO}_{1,3}^e$ on $\mathbb{R}_{1,3}$.

and since³ $\bigwedge TM \hookrightarrow \mathcal{C}\ell(M, \mathbf{g})$, sections of $\mathcal{C}\ell(M, \mathbf{g})$ (the Clifford fields) can be represented as a sum of non homogeneous differential forms.

Next (using that M is parallelizable) we introduce the global tetrad basis $\{e_\alpha\}$ on TM and in T^*M the cotetrad basis on $\{\gamma^\alpha\}$, which are dual basis. We introduce the reciprocal basis $\{e^\alpha\}$ and $\{\gamma_\alpha\}$ of $\{e_\alpha\}$ and $\{\gamma^\alpha\}$ satisfying

$$g(e_\alpha, e^\beta) = \delta_\alpha^\beta, \quad g(\gamma^\beta, \gamma_\alpha) = \delta_\alpha^\beta. \quad (4)$$

Moreover, recall that⁴

$$g = \eta_{\alpha\beta} \gamma^\alpha \otimes \gamma^\beta = \eta^{\alpha\beta} \gamma_\alpha \otimes \gamma_\beta, \quad g = \eta^{\alpha\beta} e_\alpha \otimes e_\beta = \eta_{\alpha\beta} e^\alpha \otimes e^\beta. \quad (5)$$

To present our results on the Lie derivatives of spinor fields we need to recall some other definitions, which serve also to fix our notation:

Recalling that $\text{Spin}_{1,3}^e \hookrightarrow \mathbb{R}_{1,3}^0$, we give:

Definition 4 *The left (respectively right) real spin-Clifford bundle of the spin manifold M is the vector bundle $\mathcal{C}\ell_{\text{Spin}}^l(M, \mathbf{g}) = P_{\text{Spin}_{1,3}^e}(M, \mathbf{g}) \times_l \mathbb{R}_{1,3}$ (respectively $\mathcal{C}\ell_{\text{Spin}}^r(M, \mathbf{g}) = P_{\text{Spin}_{1,3}^e}(M, \mathbf{g}) \times_r \mathbb{R}_{1,3}$) where l is the representation of $\text{Spin}_{1,3}^e$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$ (respectively, where r is the representation of $\text{Spin}_{1,3}^e$ on $\mathbb{R}_{1,3}$ given by $r(a)x = xa^{-1}$). Sections of $\mathcal{C}\ell_{\text{Spin}}^l(M, \mathbf{g})$ are called left spin-Clifford fields (respectively right spin-Clifford fields).*

Definition 5 *Let $e^l \in \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathbf{g})$ be a primitive global idempotent⁵, respectively $e^r \in \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathbf{g})$, and let $I^l(M, \mathbf{g})$ and $I^r(M, \mathbf{g})$ be the subbundles of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathbf{g})$ and $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathbf{g})$ generated by these idempotents, that is, if Ψ is a section of $I^l(M, \mathbf{g}) \subset \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathbf{g})$, and Φ is a section of $I^r(M, \mathbf{g}) \subset \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathbf{g})$, we have*

$$\Psi e^l = \Psi, \quad e^r \Phi = \Phi. \quad (6)$$

A section Ψ of $I^l(M, \mathbf{g})$ (respectively Φ of $I^r(M, \mathbf{g})$) is called a left (respectively right) ideal algebraic spinor field.

Definition 6 *A Dirac-Hestenes spinor field (DHSF) associated with Ψ is a section⁶ Ψ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^{0l}(M, \mathbf{g}) \subset \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathbf{g})$ (respectively a section Φ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^{0r}(M, \mathbf{g}) \subset \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathbf{g})$) such that⁷*

$$\Psi = \Psi e^l, \quad \Phi = e^r \Phi. \quad (7)$$

³Given the objects A and B , $A \hookrightarrow B$ means as usual that A is embedded in B and moreover, $A \subseteq B$. In particular, recall that there is a canonical vector space isomorphism between $\bigwedge \mathbb{R}^{1,3}$ and $\mathbb{R}_{1,3}$, which is written $\bigwedge \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Details in [5, 21].

⁴Where the matrix with entries $\eta_{\alpha\beta}$ (or $\eta^{\alpha\beta}$) is the diagonal matrix $(1, -1, -1, -1)$.

⁵We know that global primitive idempotents exist because M is parallelizable.

⁶ $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^{0l}(M, \mathbf{g})$ (respectively $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^{0r}(M, \mathbf{g})$) denotes the even subbundle of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathbf{g})$ (respectively $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathbf{g})$).

⁷For any Ψ the DHSF always exist, see [29].

Definition 7 *There are natural pairings:*

$$\sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}) \times \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}) \rightarrow \sec \mathcal{C}\ell(M, \mathfrak{g}), \quad (8)$$

$$\sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}) \times \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}) \rightarrow \mathcal{F}(M, \mathbb{R}_{1,3}), \quad (9)$$

such that given a section α of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ and a section β of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$ and selecting representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ ($p \in \pi^{-1}(x)$) it is

$$(\alpha\beta) := [(p; ab)] \in \mathcal{C}\ell(M, \mathfrak{g}), \quad (10)$$

$$(\beta\alpha)(x) := ba \in \mathbb{R}_{1,3}. \quad (11)$$

If alternative representatives (pu^{-1}, ua) and (pu^{-1}, bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$ we have $[(pu^{-1}; uabu^{-1})]$, that, by 3, represents the same element on $\mathcal{C}\ell(M, \mathfrak{g})$, and $(bu^{-1}ua) = ba$; thus $(\alpha\beta)(x)$ and $(\beta\alpha)(x)$ are a well defined. Following the same procedure we could define the actions [29]:

$$\sec \mathcal{C}\ell(M, \mathfrak{g}) \times \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}) \rightarrow \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}), \quad (12)$$

$$\sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}) \times \sec \mathcal{C}\ell(M, \mathfrak{g}) \rightarrow \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}), \quad (13)$$

$$\sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}) \times \mathbb{R}_{1,3} \rightarrow \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}), \quad (14)$$

$$\mathbb{R}_{1,3} \times \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}) \rightarrow \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g}). \quad (15)$$

Given a local trivialization of $\mathcal{C}\ell(M, \mathfrak{g})$ (or $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$, $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$) ($\mathcal{U} \subset M$)

$$\phi_U : \pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times \mathbb{R}_{1,3}, \quad (16)$$

we can define a local unit section by $\mathbf{1}_U(x) = \phi_U^{-1}(x, 1)$. For $\mathcal{C}\ell(M, \mathfrak{g})$, it is easy to show that a global unit section always exist, independently of the fact that M is parallizable or not. For the bundles $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^l(M, \mathfrak{g})$, $\mathcal{C}\ell_{\text{Spin}_{p,q}^e}^r(M, \mathfrak{g})$ ($\dim M = p + q$) there exist a global unit sections if, and only if, $P_{\text{Spin}_{p,q}^e}(M, \mathfrak{g})$ is trivial [28, 29]. In our case we know, by Geroch theorem, that M is parallelizable and we can define global unit sections on $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ and $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$.

Let Ξ_u be a section of $P_{\text{Spin}_{1,3}^e}(M, \mathfrak{g})$, i.e., a spin frame. We recall, in order to fix notations, that sections of $\mathcal{C}\ell(M, \mathfrak{g})$ $I^l(M, \mathfrak{g})$, $I^r(M, \mathfrak{g})$ $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$, $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$ are, respectively, the equivalence classes

$$\begin{aligned} \mathbf{C} &= [(\Xi_u, \mathcal{C}_{\Xi_u})], \\ \Psi &= [(\Xi_u, \Psi_{\Xi_u})] \quad \Phi = [(\Xi_u, \Phi_{\Xi_u})], \quad \Psi = [(\Xi_u, \Psi_{\Xi_u})], \quad \Phi = [(\Xi_u, \Phi_{\Xi_u})]. \end{aligned} \quad (17)$$

Remark 8 *When convenient, we will write $\mathcal{C}_{\Xi_u} \in \sec \mathcal{C}\ell(M, \mathfrak{g})$ to mean that there exists a section \mathbf{C} of the Clifford bundle $\mathcal{C}\ell(M, \mathfrak{g})$ defined by $[(\Xi_u, \mathcal{C}_{\Xi_u})]$. Analogous notations will be used for sections of the other bundles introduced above. Also, when there is no chance of confusion on the chosen spinor frame, we will write \mathcal{C}_{Ξ_u} simply as \mathcal{C} .*

For each spin frame, say Ξ_0 , let $\mathbf{1}_{\Xi_0}^l$ and $\mathbf{1}_{\Xi_0}^r$ be the global unit sections of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ and $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$, given by

$$\mathbf{1}_{\Xi_0}^r := [(\Xi_0, 1)], \quad \mathbf{1}_{\Xi_0}^l := [(\Xi_0, 1)]. \quad (18)$$

Remark 9 Before proceeding note that given another spin frame $\Xi_u = \Xi_0 u$, where $u : M \rightarrow \text{Spin}_{1,3}^e \subset \mathbb{R}_{1,3}^0 \subset \mathbb{R}_{1,3}$ we define the sections $\mathbf{1}_{\Xi_u}^r$ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$ and $\mathbf{1}_{\Xi_u}^l$ of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ by

$$\mathbf{1}_{\Xi_u}^r := [(\Xi_u, 1)], \quad \mathbf{1}_{\Xi_u}^l := [(\Xi_u, 1)]. \quad (19)$$

It has been proved in [28, 29] that the relation between $\mathbf{1}_{\Xi}^r$ and $\mathbf{1}_{\Xi_0}^r$ and between $\mathbf{1}_{\Xi}^l$ and $\mathbf{1}_{\Xi_0}^l$ are given by

$$\mathbf{1}_{\Xi_u}^r = u^{-1} \mathbf{1}_{\Xi_0}^r = \mathbf{1}_{\Xi_0}^r U^{-1}, \quad \mathbf{1}_{\Xi_u}^l = U \mathbf{1}_{\Xi_0}^l = \mathbf{1}_{\Xi_0}^l u \quad (20)$$

where U is the section of $\mathcal{C}\ell(M, \mathfrak{g})$ defined by the equivalence class

$$U = [(\Xi_0, u)]. \quad (21)$$

The unity sections $\mathbf{1}_{\Xi_u}^l$ and $\mathbf{1}_{\Xi_u}^r$ satisfies the important relations⁸

$$\mathbf{1}_{\Xi_u}^l \mathbf{1}_{\Xi_u}^r = 1 \in \sec \mathcal{C}\ell(M, \mathfrak{g}), \quad \mathbf{1}_{\Xi_u}^r \mathbf{1}_{\Xi_u}^l = 1 \in \mathcal{F}(M, \mathbb{R}_{1,3}), \quad (22)$$

Definition 10 A representative of a DHSF Ψ (respectively Φ) in the Clifford bundle $\mathcal{C}\ell(M, \mathfrak{g})$ relative to a spin frame Ξ_u is a section $\psi_{\Xi_u} = [(\Xi_u, \psi_{\Xi_u})]$ of $\mathcal{C}\ell^0(M, \mathfrak{g})$ (respectively, a section $\phi_{\Xi_u} = [(\Xi_u, \phi_{\Xi_u})]$ of $\mathcal{C}\ell^0(M, \mathfrak{g})$) given by [28, 29]

$$\psi_{\Xi_u} = \Psi \mathbf{1}_{\Xi_u}^r, \quad \mathbf{1}_{\Xi_u}^l \Phi = \phi_{\Xi_u}. \quad (23)$$

Representatives in the Clifford bundle of Ψ relative to spin frames, say $\Xi_{u'}$ and Ξ_u , are related by⁹

$$\psi_{\Xi_{u'}} U'^{-1} = \psi_{\Xi_u} U^{-1}. \quad (24)$$

Also, representatives in the Clifford bundle of Φ relative to spin frames $\Xi_{u'}$ and Ξ_u are related by

$$U' \phi_{\Xi_{u'}} = U \phi_{\Xi_u}. \quad (25)$$

⁸ $\mathcal{C}\ell^0(M, \mathfrak{g})$ denotes the even subbundle of $\mathcal{C}\ell(M, \mathfrak{g})$.

⁹This relation has been used in [27] to define a DHSF as an appropriate equivalence class of even sections of the Clifford bundle $\mathcal{C}\ell(M, \mathfrak{g})$.

3 On the Concept of Lie Derivatives of Spinor Fields in Lorentzian Manifolds

Lie derivatives of tensor fields are defined once we give the concept of the push forward and pullback mappings (which serves the purpose of defining the image of the tensor field) associated to one-parameter groups of diffeomorphisms generated by vector fields. These concepts are well known and very important in the derivation of conserved currents in physical theories.

It happens that physical theories need also the concept of spinor fields living on a Lorentzian manifold and the question arises as how to define a meaningful image for these objects under a diffeomorphism. There are a lot of different approaches to the subject, as the reader can learn consulting, e.g., [1, 2, 3, 5, 6, 8, 9, 11, 12, 13, 14, 17, 18, 19, 20, 22, 26, 31]. In what follows using the definition of left (and right) *real* spinor fields (in particular, Dirac-Hestenes spinor fields) [28, 25, 29] living in a Lorentzian manifold and their representatives in the Clifford bundle we give a geometrical motivated definition for their images and a corresponding definition of the Lie derivative for spinor and Clifford fields. We compare our definition with some others appearing in the literature.

We already recalled that fixing a global spinor basis¹⁰ $\Xi_0(x) = (x, u_0(x))$ for $P_{\text{Spin}_{1,3}^e}(M, \mathbf{g})$, and given an algebraic spinor Ψ , the associated DHSF Ψ can be represented in the Clifford bundle by the object¹¹

$$\psi_{\Xi_0} \in \sec \mathcal{C}\ell^0(M, \mathbf{g}). \quad (26)$$

Remark 11 When $\psi_{\Xi_0} \tilde{\psi}_{\Xi_0} \neq 0$ we can easily show that ψ_{Ξ_0} has the following representation¹²

$$\psi_{\Xi_0} = \rho^{\frac{1}{2}} e^{-\frac{\tau_{\mathbf{g}} \beta}{2}} R, \quad (27)$$

where $\rho, \beta \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathbf{g})$ and [23]

$$R = e^{\mathcal{F}} \in \sec \text{Spin}_{1,3}^e(M, \mathbf{g}) \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathbf{g}), \quad (28)$$

with $\mathcal{F} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathbf{g})$.

Let $\xi \in \sec TM$ be a smooth vector field. For any $x \in M$ there exists a unique integral curve of ξ , given by $t \mapsto h(t, x)$, with $x = h(0, x)$. We recall that for $(t, x) \in I(x) \times \mathbb{M}$ ($I(x) \subset \mathbb{R}$) the mapping $h: (t, x) \mapsto h(t, x)$ is called the flow of ξ . We suppose in what follows that the mappings $h_t := h(t, \cdot) : M \rightarrow M$,

¹⁰Such a basis must exist according to Geroch Theorem (3) [10].

¹¹The notation $\mathbf{P} \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$ means that there exists a section s_P of $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathbf{g})$ given by or any $x \in M$ by the equivalence class $[(\Xi_0(x), \mathbf{P}(x))]$ where $\mathbf{P} : M \rightarrow \bigwedge^p \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$.

¹²The product $\rho(x)^{\frac{1}{2}} e^{-\frac{\tau_{\mathbf{g}}(x)\beta(x)}{2}} R(x)$ in Eq.(27) must be understood as meaning the product $[(\Xi_0(x), \rho(x))][(\Xi_0(x), e^{-\frac{\tau_{\mathbf{g}}(x)\beta(x)}{2}})] [(\Xi_0(x), R(x))] = [(\Xi_0(x), \rho^{\frac{1}{2}} e^{-\frac{\tau_{\mathbf{g}}\beta}{2}} R)]$, where $\rho(x), \beta(x) \in \bigwedge^0 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, $R(x) \in \text{Spin}_{1,3}^e \subset \mathbb{R}_{1,3}^0$ and $\tau_{\mathbf{g}}(x) \in \bigwedge^4 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$.

$x \mapsto x' = h_t(x)$ generate a one-parameter group of diffeomorphisms of M (i.e., $I(x) = \mathbb{R}$) We have (for a fixed $x \in M$)

$$\xi(h(t, x)) = \left. \frac{d}{dt} h(t, x) \right|_{t=0}. \quad (29)$$

Now, recall that ψ_{Ξ_0} determines global 1-form fields on M , namely $V^\alpha \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathfrak{g})$ such that at $x' = h_t(x)$

$$\begin{aligned} V^\alpha(x') &= \psi_{\Xi_0}(x') \gamma^\alpha(x') \tilde{\psi}_{\Xi_0}(x') = \rho(x') R(x') \gamma^\alpha(x') \tilde{R}(x') \\ &= \rho(x') L_\beta^\alpha(x') \gamma^\beta(x') = \rho(x') \Gamma^\alpha(x'). \end{aligned} \quad (30)$$

The *pullbacks* of γ^α , Γ^α and V^α are the fields

$$\gamma_t'^\alpha = h_t^* \gamma^\alpha, \quad \Gamma_t'^\alpha = h_t^* \Gamma^\alpha, \quad V_t'^\alpha = h_t^* V^\alpha. \quad (31)$$

At $x \in M$, expressing the diffeomorphism $x' = h_t(x)$ in a local coordinate chart covering $\mathcal{U} \subset M$ it is

$$\gamma_t'^\alpha(x) = h_{t\mu}^\alpha(x'(x)) \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \quad (32)$$

with similar formulas for the $\Gamma_t'^\alpha$ and $V_t'^\alpha$. In particular take notice that

$$\begin{aligned} V_t'^\alpha(x) &= \rho(x'(x)) L_\beta^\alpha(x'(x)) \gamma_t'^\beta(x) \\ &= \rho(x'(x)) R'(x) \gamma'^\alpha(x) \tilde{R}'(x). \end{aligned} \quad (33)$$

Remark 12 *It is clear that $\{\gamma_t'^\alpha(x)\}, \{\Gamma_t'^\alpha(x)\}$ are not orthonormal basis for $T_x M$ relative to g_x unless ξ is a Killing vector field, but of course, they are orthonormal basis of $T_x M$ relative to $g'_x = h_t^* g|_x$.*

Eq.(28) says that $R \in \sec \text{Spin}_{1,3}^e \subset \sec \mathcal{C}\ell^0(M, \mathfrak{g})$ is the exponential of a biform field [23], say $R = e^{\mathcal{F}(x)}$ Thus, we see that there exists *no* difficulty in defining the pullback¹³ of $\rho^{\frac{1}{2}} e^{-\frac{\tau g \beta}{2}} e^{\mathcal{F}(x)}$ under h_t (or of more generally, for any $\psi_{\Xi_0} \in \mathcal{C}\ell^0(M, \mathfrak{g})$), which will be written as

$$\rho^{\frac{1}{2}}(x'(x)) e^{-\frac{\tau' g t(x) \beta(x'(x))}{2}} e^{\mathcal{F}'_t(x)}. \quad (34)$$

However, we immediately have a

Problem: The object defined by Eq.(34) is of course, a representative in $\mathcal{C}\ell^0(M, g)$ of some Dirac-Hestenes spinor field but there is no way to know to which the spinor frame that object is associated.

¹³Or of more generally, even for a non invertible $\psi_{\Xi_0} \in \mathcal{C}\ell^0(M, \mathfrak{g})$, which is a sum of even nonhomogeneous differetial forms.

Thus, we must find another way to define the Lie derivative for spinor fields. Our way, as we will see, is based in a geometric motivated definition for the concept of image of Clifford and spinor fields under diffeomorphisms generated by one-parameter group associated to an arbitrary vector field ξ . But, we need first to introduce some results, starting with the

Proposition 13 *Let \mathcal{L}_ξ denotes the standard Lie derivative of tensor fields. If ξ is a Killing vector field then*

$$\mathcal{L}_\xi \gamma^\alpha = \frac{1}{4}[L(\xi) + d\xi, \gamma^\alpha] \quad (35)$$

$$= D_\xi \gamma^\alpha + \frac{1}{4}[d\xi, \gamma^\alpha] \quad (36)$$

with

$$L(\xi) := \frac{1}{2}(c_{\alpha\kappa\iota} + c_{\kappa\alpha\iota} + c_{\iota\alpha\kappa})\xi^\kappa \gamma^\alpha \wedge \gamma^\iota \quad (37)$$

where $c_{\kappa\iota}^{\alpha\cdot\cdot}$ are the structure coefficients of the basis $\{e_\alpha\}$ dual of $\{\gamma^\alpha\}$.

Proof. First recall that Eq.(36) is clearly equal to Eq.(35) since once $D_{e_\kappa} \gamma^\alpha = -\omega_{\kappa\iota}^{\alpha\cdot\cdot} \gamma^\iota$ it is

$$\omega_{\alpha\kappa\iota} = \frac{1}{2}(c_{\alpha\kappa\iota} + c_{\kappa\alpha\iota} + c_{\iota\alpha\kappa}). \quad (38)$$

It follows that

$$L(\xi) = 2\omega_\xi \quad (39)$$

where

$$\omega_\xi = \frac{1}{2}\xi^\kappa \omega_{\alpha\kappa\iota} \gamma^\alpha \wedge \gamma^\iota \quad (40)$$

is the “connection biform” and so

$$\frac{1}{4}[L(\xi), \gamma^\alpha] = \frac{1}{2}[\omega_\xi, \gamma^\alpha] = -\gamma^\alpha \lrcorner \omega_\xi = D_\xi \gamma^\alpha. \quad (41)$$

Now, recalling Cartan’s magical formula, and the following identities

$$\mathcal{L}_\xi \gamma^\alpha = \xi \lrcorner d\gamma^\alpha + d(\xi \lrcorner \gamma^\alpha) = \xi \lrcorner d\gamma^\alpha + d(\gamma^\alpha \lrcorner \xi) \quad (42)$$

$$d = \gamma^\iota \wedge D_{e_\iota}, \quad (43)$$

we have

$$\begin{aligned} \mathcal{L}_\xi \gamma^\alpha &= \xi \lrcorner (\gamma^\iota \wedge D_{e_\iota} \gamma^\alpha) + d(\xi^\alpha) \\ &= (\xi \lrcorner \gamma^\iota) D_{e_\iota} \gamma^\alpha - (\xi \lrcorner D_{e_\iota} \gamma^\alpha) \gamma^\iota + e_\iota(\xi^\alpha) \gamma^\iota \\ &= \xi^\iota D_{e_\iota} \gamma^\alpha + (\xi^\kappa \gamma_{\kappa\iota} \lrcorner \omega_{\cdot\iota\lambda}^{\alpha\cdot\cdot} \gamma^\lambda) \gamma^\iota + e_\iota(\xi^\alpha) \gamma^\iota \\ &= D_\xi \gamma^\alpha + [\xi^\lambda \omega_{\cdot\iota\lambda}^{\alpha\cdot\cdot}] \gamma^\iota + e_\iota(\xi^\alpha) \gamma^\iota \\ &= D_\xi \gamma^\alpha + (D_\iota \xi^\alpha) \gamma^\iota \end{aligned}$$

We get,

$$\mathcal{L}_\xi \gamma^\alpha = D_\xi \gamma^\alpha + (D_\iota \xi^\alpha) \gamma^\iota. \quad (44)$$

Now, it remains to show that for ξ a Killing vector field $\frac{1}{4}[d\xi, \gamma^\alpha]$ is equal to $(D_\iota \xi^\alpha) \gamma^\iota$. We have

$$\begin{aligned} \frac{1}{4}[d\xi, \gamma^\alpha] &= -\frac{1}{2}\gamma^\alpha \lrcorner d\xi \\ &= -\frac{1}{2}\gamma^\alpha \lrcorner (\gamma^\iota \wedge D_{e_\iota} \xi) = -\frac{1}{2}\gamma^\alpha \lrcorner \{\gamma^\iota \wedge (D_\iota \xi_\kappa) \gamma^\kappa\} \\ &= \frac{1}{2} D_\iota \xi_\kappa \gamma^\alpha \lrcorner \{\gamma^\kappa \wedge \gamma^\iota\} = D_\iota \xi^\alpha \gamma^\iota - \frac{1}{2} (D_\iota \xi_\kappa + D_\kappa \xi_\iota) \eta^{\alpha\iota} \gamma^\kappa = D_\iota \xi^\alpha \gamma^\iota \end{aligned} \quad (45)$$

and the proposition is proved. ■

Remark 14 We emphasize that in the derivation of the previous result it has not been used the fact that \mathcal{L}_ξ must be a derivation in the Clifford bundle, since all operations done requires only simple and well known formulas from the calculus of differential forms.

Remark 15 Moreover, one can easily show using the previous results that when $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathfrak{g})$ and $\xi \in \sec TM$ is a Killing vector field then

$$\mathcal{L}_\xi \mathcal{C} = \mathfrak{d}_\xi \mathcal{C} + \frac{1}{4}[\mathbf{S}(\xi), \mathcal{C}]. \quad (46)$$

Indeed, Eq.(46) follows trivially by induction and noting that $\mathcal{L}_\xi(\mathcal{A}\mathcal{B}) = \mathcal{L}_\xi(\mathcal{A})\mathcal{B} + \mathcal{A}\mathcal{L}_\xi(\mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \sec \mathcal{C}\ell(M, \mathfrak{g})$, when $\xi \in \sec TM$ is a Killing vector field.

This suggests that $L(\xi)$ should be involved in the definition of the Lie derivative of spinor fields. Based on this, and recalling Eq.(28) we propose that the *spinor lifting* of an integral curve of a generic smooth vector field $\xi \in \sec TM$ to $P_{\text{Spin}^e_{1,3}}(M, \mathfrak{g})$ in the parallelizable manifold M equipped with the global orthonormal cobasis $\{\gamma^\alpha\}$ is given by the following

Definition 16 Consider the integral curve $h_t : \mathbb{R} \rightarrow M$ of an arbitrary smooth vector field ξ . The spinor lifting \check{h}_t of h_t to $P_{\text{Spin}^e_{1,3}}(M, \mathfrak{g})$ is the curve

$$\check{h}_t(p) = (h_t(\pi(p)), au_t(h_t(\pi(p)))) \quad (47)$$

$$u_t(x) := e^{-\frac{1}{4}t(\mathbf{S}(\xi)(x))} \in \text{Spin}^e_{1,3}, \quad (48)$$

$$\mathbf{S}(\xi) = L(\xi) + d\xi, \quad (49)$$

with $\pi(p) = \pi((x, a)) = x$.

To see why the above definition is really important consider that for $t \ll 1$ it is

$$u_t = 1 - \frac{1}{4}t\mathbf{S}(\xi) + O(t^2) + \dots \quad (50)$$

Then, we have for $t \ll 1$ that

$$\begin{aligned} u_t^{-1} \gamma^\alpha u_t &= \{1 + \frac{1}{4} t \mathbf{S}(\xi) + O(t^2) + \dots\} \gamma^\alpha \{1 - \frac{1}{4} t \mathbf{S}(\xi) + O(t^2) + \dots\} \\ &= \gamma^\alpha + \frac{1}{4} t [\mathbf{S}(\xi), \gamma^\alpha] + O(t^2) + \dots \end{aligned} \quad (51)$$

Deriving in $t = 0$ we obtain the expression of the previous proposition.

Now, recall that the pullback $\gamma_t'^\alpha = h_t^* \gamma^\alpha$ when ξ is any (arbitrary) vector field for $t \ll 1$ is

$$\gamma_t'^\alpha(x) = \gamma^\alpha(x) + t \mathcal{L}_\xi \gamma^\alpha(x) + O(t^2) + \dots \quad (52)$$

Using the Proposition (13) (valid when ξ is Killing), comparing Eq.(52) with Eq.(51) and recalling Eq.(35), we see that up to the *first order* we have for this case

$$\gamma_t'^\alpha(x) = u_t^{-1}(x) \gamma^\alpha(x) u_t(x) \quad (53)$$

Remark 17 We could write the first member of Eq.(53) and keeping terms up to first order as

$$\gamma_t'^\alpha = \Lambda_{t\beta}^\alpha \gamma^\beta = (\delta_\beta^\alpha + t \Sigma_\beta^\alpha) \gamma^\beta \quad (54)$$

where $\Lambda_{t\beta}^\alpha(x) \in \text{SO}_{1,3}^e$ for any $x \in \mathcal{U} \subset M$. Thus

$$\gamma^\beta \Sigma_\beta^\alpha = \frac{1}{4} [\mathbf{S}(\xi), \gamma^\alpha] = -\frac{1}{2} \gamma^\alpha \lrcorner (\mathbf{S}(\xi)) \quad (55)$$

and then

$$\begin{aligned} \Sigma_\kappa^\alpha &= \gamma_\kappa \lrcorner \gamma^\beta \Sigma_{t\beta}^\alpha = -\frac{1}{2} \gamma_\kappa \lrcorner (\gamma^\alpha \lrcorner \mathbf{S}(\xi)) \\ &= -\frac{1}{2} (\gamma_\kappa \wedge \gamma^\alpha) \lrcorner \mathbf{S}(\xi) \\ &= \frac{1}{2} (\gamma^\alpha \wedge \gamma_\kappa) \cdot \mathbf{S}(\xi) = \frac{1}{2} \mathbf{S}(\xi) \cdot (\gamma^\alpha \wedge \gamma_\kappa). \end{aligned} \quad (56)$$

From Eq.(53), the Lie derivative (when ξ is Killing) $\mathcal{L}_\xi \gamma^\alpha$ can be calculated in two ways, using the usual definition by pullback or by the action of u_t . Note, however that the action of u_t is always orthogonal, regardless of ξ be Killing. We will use this fact to give our geometric motivated concept of Lie derivatives for Clifford and spinor fields.

Remark 18 It is very important to keep in mind that although the usual Lie derivative of γ^α in the direction of an arbitrary smooth Killing vector field ξ is given by Eq.(53) this does not implies, of course that $\mathcal{L}_\xi \mathbf{g}$ is null for an arbitrary vector field. In fact $\mathcal{L}_\xi \mathbf{g} = 0$ defines a Killing vector field and $\mathcal{L}_\xi \mathbf{g} = 0$ implies that there exists $U_t \in \text{sec } P_{\text{Spin}_{1,3}^e}(M, \mathbf{g})$ such that in all orders in t (not only in first order as in Eq.(53)) it holds that

$$h_t^* \gamma^\alpha = U_t^{-1} \gamma^\alpha U_t = \Lambda_{t\beta}^\alpha \gamma^\beta \quad (57)$$

where for all $x \in M$, $\Lambda_{t\beta}^\alpha(x) \in \text{SO}_{1,3}^e$.

4 The Spinor Lie Derivative $\overset{s}{\mathcal{L}}_{\xi}$

4.1 Spinor Images of Clifford and Spinor Fields

Given the spinorial frame $\Xi_{u_t}(x) = (x, u_t)$ in $P_{\text{Spin}_{1,3}^e}(M, \mathfrak{g})$ we see that the basis $\{\check{\gamma}_t^\alpha\}$ of $P_{\text{SO}_{1,3}^e}(M, \mathfrak{g})$ such that

$$\check{\gamma}_t^\alpha(x) = u_t^{-1}(x)\gamma^\alpha(x)u_t(x) = \Lambda_{t\beta}^\alpha(x)\gamma^\beta(x), \quad (58)$$

is always orthonormal relative to \mathfrak{g} . This suggests to define a mapping ${}^s\mathbf{h}_t$ (associated with a one parameter group of diffeomorphisms \mathbf{h}_t generated by a vector field ξ acting on sections $\bigwedge^p T^*M, \mathcal{C}\ell(M, \mathfrak{g}), \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}), \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$. With $x' = \mathbf{h}_t(x)$ we start giving

Definition 19

$$\begin{aligned} {}^s\mathbf{h}_t : \sec \mathcal{C}\ell(M, \mathfrak{g}) &\leftarrow \sec \bigwedge^p T^*M \rightarrow \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathfrak{g}), \\ P(x') &\mapsto \check{P}_t(x) = \frac{1}{p!} P_{i_1 \dots i_p}(x'(x)) \check{\gamma}_t^{i_1}(x) \cdots \check{\gamma}_t^{i_p}(x) \\ &= \frac{1}{p!} P_{i_1 \dots i_p}(x'(x)) u_t^{-1} \gamma^{i_1}(x) \cdots \gamma^{i_p}(x) u_t \end{aligned} \quad (59)$$

with

$$\begin{aligned} P(x) &= \frac{1}{p!} P_{i_1 \dots i_p}(x) \gamma^{i_1}(x) \cdots \gamma^{i_p}(x) \neq \check{P}_t(x) \\ P(x') &= \frac{1}{p!} P_{i_1 \dots i_p}(x') \gamma^{i_1}(x') \cdots \gamma^{i_p}(x') \end{aligned} \quad (60)$$

Eq.(59) extends by linearity to all sections of $\mathcal{C}\ell(M, \mathfrak{g})$. Given any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathfrak{g})$ we will call $\check{\mathcal{C}}_t$ the spinor image of \mathcal{C} .

4.2 Spinor Derivative of Clifford and Spinor Fields

Definition 20 The spinor Lie Derivative $\overset{s}{\mathcal{L}}_{\xi}$ of a Clifford field $\mathbf{C} = [(\Xi_0, \mathcal{C})]$ (a section of $\mathcal{C}\ell(M, \mathfrak{g})$) in the direction of an arbitrary vector field ξ is

$$\overset{s}{\mathcal{L}}_{\xi} \mathbf{C} = \left. \frac{d}{dt} \check{\mathcal{C}}_t \right|_{t=0} \quad (61)$$

A trivial calculation gives

$$\overset{s}{\mathcal{L}}_{\xi} \mathbf{C} = \mathfrak{d}_{\xi} \mathbf{C} + \frac{1}{4} [S(\xi), \mathbf{C}]. \quad (62)$$

Given that a left DHSF Ψ , a section of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ (respectively Φ , a section of $\mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$) can be written as

$$\psi_{\Xi_0} = \Psi \mathbf{1}_{\Xi_0}^r, \quad \phi_{\Xi_0} = \mathbf{1}_{\Xi_0}^l \Phi, \quad (63)$$

$$\psi_{\Xi_0} \mathbf{1}_{\Xi_0}^l = \Psi \mathbf{1}_{\Xi_0}^r \mathbf{1}_{\Xi_0}^l = \Psi \mathbf{1}, \quad \mathbf{1}_{\Xi_0}^r \phi_{\Xi_0} = \mathbf{1}_{\Xi_0}^r \mathbf{1}_{\Xi_0}^l \Phi = \mathbf{1} \Phi, \quad (64)$$

$$\Psi = \psi_{\Xi_0} \mathbf{1}_{\Xi_0}^l = \psi_{\Xi_u} \mathbf{1}_{\Xi_u}^l, \quad \Phi = \mathbf{1}_{\Xi_0}^r \phi_{\Xi_0} = \mathbf{1}_{\Xi_u}^r \phi_{\Xi_u}. \quad (65)$$

Using Eq.(65) and thar $\psi_{\Xi_0}, \phi_{\Xi_0} \in \sec \mathcal{C}\ell^0(M, \mathfrak{g})$, where we know how to act, we propose the following definition:

Definition 21 *The spinor images of Ψ and Φ are:*

$${}^s h_t \Psi \circ h_t x = {}^s \Psi_t(x) := (\mathbf{u}_t^{-1} \psi_{\Xi_0}(x) \mathbf{u}_t) {}^s \mathbf{1}_{t\Xi_0}^l, \quad (66)$$

$${}^s h_t \Phi \circ h_t x = {}^s \Phi_t(x) := {}^s \mathbf{1}_{t\Xi_0}^r (\mathbf{u}_t^{-1} \phi_{\Xi_0}(x) \mathbf{u}_t) \quad (67)$$

and

$${}^s \mathbf{1}_{t\Xi_0}^l := \mathbf{u}_t^{-1} \mathbf{1}_{\Xi_0}^l, \quad {}^s \mathbf{1}_{t\Xi_0}^r = \mathbf{1}_{\Xi_0}^r \mathbf{u}_t. \quad (68)$$

Definition 22 *With these actions, we define:*

$$\begin{aligned} \mathcal{L}_\xi \Psi &:= \frac{d}{dt} {}^s \Psi_t(x)|_{t=0}, \\ \mathcal{L}_\xi \Phi &:= \frac{d}{dt} {}^s \Phi_t(x)|_{t=0} \end{aligned} \quad (69)$$

The objects ${}^s \Psi_t, {}^s \Psi_t, {}^s \psi_{t\Xi_0}, {}^s \Phi_t, {}^s \Phi_t, {}^s \phi_{t\Xi_0}, {}^s C_t$ (sections of $I^l(M, \mathfrak{g}), I^r(M, \mathfrak{g}), \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g}), \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^r(M, \mathfrak{g})$) will be referred in what follows as the *spinor images* of the fields $\Psi, \psi, \psi_{\Xi_0}, \Phi, \phi, \phi_{\Xi_0}, C$.

A trivial calculation gives

$$\begin{aligned} \mathcal{L}_\xi \Psi &= \mathfrak{d}_\xi \Psi + \frac{1}{4} \mathbf{S}(\xi) \Psi, \\ \mathcal{L}_\xi \Phi &= \mathfrak{d}_\xi \Phi - \Phi \frac{1}{4} \mathbf{S}(\xi) \end{aligned} \quad (70)$$

and we observe that it is

$$\mathcal{L}_\xi \mathbf{1}_{\Xi_0}^l = \frac{1}{4} \mathbf{S}(\xi) \mathbf{1}_{\Xi_0}^l, \quad \mathcal{L}_\xi \mathbf{1}_{\Xi_0}^r = -\frac{1}{4} \mathbf{1}_{\Xi_0}^r \mathbf{S}(\xi). \quad (71)$$

Remark 23 *In the Clifford bundle in the basis Ξ_0 , $\psi_{\Xi_0} \in \sec \mathcal{C}\ell(M, \mathfrak{g})$ is the representative of $\Psi \in \sec \mathcal{C}\ell_{\text{Spin}_{1,3}^e}^l(M, \mathfrak{g})$ and if we calculated its spinor Lie derivative as a section of $\mathcal{C}\ell(M, \mathfrak{g})$ we should get, of course*

$$\mathcal{L}_\xi \psi_{\Xi_0} = \mathfrak{d}_\xi \psi_{\Xi_0} + \frac{1}{4} [L(\xi) + d\xi, \psi_{\Xi_0}]. \quad (72)$$

This does not mimics the spinor Lie derivative of a DHSF Ψ . Since one of the main reasons to introduce representatives in the Clifford bundle of Dirac-Hestenes spinor fields is to have an easy computation tool when using these representatives together with other Clifford fields we will agree to take as the Lie derivative of ψ_{Ξ_0} an effective Lie derivative denoted $\overset{(s)}{\mathcal{L}}_{\xi}\psi_{\Xi_0}$ where the pullback of ψ_{Ξ_0} is the formula given by Eq.(34). Thus,

$$\overset{(s)}{\mathcal{L}}_{\xi}\psi_{\Xi_0} = \mathfrak{d}_{\xi}\psi_{\Xi_0} + \frac{1}{4}L(\xi)\psi_{\Xi_0} + \frac{1}{4}d\xi\psi_{\Xi_0} \quad (73)$$

We then write for $\mathcal{C} \in \sec \mathcal{Cl}(M, \mathfrak{g})$ and ψ_{Ξ_0} as just defined

$$\overset{(s)}{\mathcal{L}}_{\xi}(\mathcal{C}\psi_{\Xi_0}) = (\overset{s}{\mathcal{L}}_{\xi}\mathcal{C})\psi_{\Xi_0} + \mathcal{C}(\overset{(s)}{\mathcal{L}}_{\xi}\psi_{\Xi_0}). \quad (74)$$

Remark 24 An analogous concept to $\overset{(s)}{\mathcal{L}}_{\xi}$ has been introduced in [29] for the covariant derivative of representatives in the Clifford bundle of Dirac-Hestenes spinor fields and we recall that for $\mathcal{C} \in \sec \mathcal{Cl}(M, \mathfrak{g})$ and ψ_{Ξ_0} as above defined we have

$$\begin{aligned} \overset{(s)}{D}_{\xi}(\mathcal{C}\psi_{\Xi_0}) &= (D_{\xi}\mathcal{C})\psi_{\Xi_0} + \mathcal{C}(\overset{(s)}{D}_{\xi}\psi_{\Xi_0}), \\ D_{\xi}\mathcal{C} &= \mathfrak{d}_{\xi}\mathcal{C} + \frac{1}{2}[\omega_{\xi}, \mathcal{C}], \\ \overset{(s)}{D}_{\xi}\psi_{\Xi_0} &= \mathfrak{d}_{\xi}\psi_{\Xi_0} + \frac{1}{2}\omega_{\xi}\psi_{\Xi_0}. \end{aligned} \quad (75)$$

with

$$\omega_{\xi} := \frac{1}{2}\xi^{\kappa}\omega_{\alpha\kappa\beta}\gamma^{\alpha}\gamma^{\beta}. \quad (76)$$

called the “connection 2-form”. Henceforth, to simplify the notation, the covariant derivative acting in a representative in the Clifford bundle of a DHSF will be written as

$$D_{\xi}^s\psi_{\Xi_0} = \mathfrak{d}_{\xi}\psi_{\Xi_0} + \frac{1}{2}\omega_{\xi}\psi_{\Xi_0}$$

and we will write also $\overset{s}{\mathcal{L}}_{\xi}\psi_{\Xi_0}$ (given by Eq.(74)) instead of $\overset{(s)}{\mathcal{L}}_{\xi}\psi_{\Xi_0}$.

Remark 25 One can easily verify that with this agreement we have a perfectly consistent formalism. Indeed, recalling that the spinor bundles are modules over $\mathcal{Cl}(M, \mathfrak{g})$ and that any section \mathbf{C} of $\mathcal{Cl}(M, \mathfrak{g})$ (see Eq.(8)) can be written as the product of a section Ψ of $\mathcal{Cl}_{\text{Spin}_{1,3}}^l(M, \mathfrak{g})$ by a section Φ of $\mathcal{Cl}_{\text{Spin}_{1,3}}^r(M, \mathfrak{g})$, i.e.,

$\mathbf{C} = \Psi\Phi$ we immediately verify that the operator $\overset{s}{\mathcal{L}}_{\xi}$ satisfies when applied to Clifford and spinor fields the Leibniz rule, i.e.,

$$\overset{s}{\mathcal{L}}_{\xi}(\Psi\Phi) = (\overset{s}{\mathcal{L}}_{\xi}\Psi)\Phi + \Psi(\overset{s}{\mathcal{L}}_{\xi}\Phi), \quad (77)$$

$$\overset{s}{\mathcal{L}}_{\xi}(\mathbf{C}\Psi) = (\overset{s}{\mathcal{L}}_{\xi}\mathbf{C})\Psi + \mathbf{C}(\overset{s}{\mathcal{L}}_{\xi}\Psi), \quad (78)$$

$$\overset{s}{\mathcal{L}}_{\xi}(\Phi\mathbf{C}) = (\overset{s}{\mathcal{L}}_{\xi}\Phi)\mathbf{C} + \Phi(\overset{s}{\mathcal{L}}_{\xi}\mathbf{C}). \quad (79)$$

5 The Spinor Lie Derivative Written in Terms of Covariant Derivatives

Recalling Eq.(39)

$$\frac{1}{4}L(\xi)\psi_{\Xi_0} = \frac{1}{2}\omega_\xi\psi_{\Xi_0} \quad (80)$$

and that

$$d\xi = \gamma^\alpha \wedge (D_{e_\alpha}\xi) = \frac{1}{2}(D_\alpha\xi_\beta - D_\beta\xi_\alpha)\gamma^\alpha\gamma^\beta \quad (81)$$

we see that Eq.(75) gives

$$\overset{s}{\mathcal{L}}_\xi\psi_{\Xi_0} = D_\xi\psi_{\Xi_0} - \frac{1}{8}(D_\alpha\xi_\beta - D_\beta\xi_\alpha)\gamma^\alpha\gamma^\beta\psi_{\Xi_0}. \quad (82)$$

5.1 Calculation of $\overset{s}{\mathcal{L}}_\xi\psi_{\Xi_0}$ in Coordinates

We show now that the spinor Lie derivative of spinor fields coincides with the one first introduced by Lichnerowicz [22]. To see this, we evaluate the spinor Lie derivative of a spinor field introducing coordinates $\{x^\mu\}$ for $\mathcal{U} \subset M$. We write

$$\gamma^\alpha = h_\mu^\alpha dx^\mu \quad (83)$$

and with D the Levi-Civita connection of g we write as usual

$$D_{\partial_\mu}\gamma^\beta = -\omega_{\mu\alpha}^{\beta\cdot\cdot}\gamma^\alpha, \quad D_{\partial_\mu}dx^\nu = -\Gamma_{\mu\tau}^{\nu\cdot\cdot}dx^\tau. \quad (84)$$

Thus, as well known

$$\partial_\mu h_\nu^\alpha + \omega_{\mu\beta}^{\alpha\cdot\cdot}h_\nu^\beta - h_\sigma^\alpha\Gamma_{\mu\cdot\nu}^{\sigma\cdot\cdot} = 0, \quad (85)$$

from where we get

$$\omega_{\alpha\mu\beta} = -(\partial_\mu h_{\alpha\nu})h_\beta^\nu + \Gamma_{\mu\alpha\beta}. \quad (86)$$

Take notice that in writing Eq.(86) we used in agreement with the original definition of the Christoffel symbols that it is *licit* to write in a coordinate basis (as some authors do, e.g., [16, 24])

$$\Gamma_{\cdot\mu\nu}^{\rho\cdot\cdot} = \Gamma_{\mu\cdot\nu}^{\rho\cdot\cdot} = g^{\rho\sigma}\Gamma_{\mu\sigma\nu} := \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right).$$

So, we get

$$\begin{aligned} \overset{s}{\mathcal{L}}_\xi\psi_{\Xi_0} &= \mathfrak{d}_\xi\psi_{\Xi_0} - \frac{1}{4}h_\beta^\nu\{\xi^\mu\partial_\mu h_{\alpha\nu} + \xi^\mu\Gamma_{\alpha\mu\beta} - (\partial_\nu\xi_\mu)h_\alpha^\mu\}\gamma^\alpha \wedge \gamma^\beta\psi_{\Xi_0} \\ &= \mathfrak{d}_\xi\psi_{\Xi_0} - \frac{1}{4}h_\beta^\nu\{\xi^\mu\partial_\mu h_{\alpha\nu} - (\partial_\nu\xi_\mu)h_\alpha^\mu\}\gamma^\alpha \wedge \gamma^\beta\psi_{\Xi_0} \\ &\quad - \frac{1}{4}\{\xi^\rho\Gamma_{\rho\mu\nu}\}dx^\mu \wedge dx^\nu\psi_{\Xi_0} \\ &= \mathfrak{d}_\xi\psi_{\Xi_0} - \frac{1}{4}h_\beta^\nu\{\xi^\mu\partial_\mu h_{\alpha\nu} - (\partial_\nu\xi_\mu)h_\alpha^\mu\}\gamma^\alpha \wedge \gamma^\beta\psi_{\Xi_0} \end{aligned} \quad (87)$$

where the last term in the second line of Eq.(87) is null because $\Gamma_{\rho\mu\nu} = \Gamma_{\rho\nu\mu}$.

5.2 Spinor Lie Derivative of Covariant Dirac Spinor Fields

Definition 26 For completeness we recall that the spinor Lie derivative of a covariant Dirac spinor field¹⁴ $\Psi \in \sec P_{\text{Spin}_{1,3}^e}(M, \mathbf{g}) \times_{\mu} \mathbb{C}^4$, with μ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $Sl(2, C) \simeq \text{Spin}_{1,3}^e$ is

$$\begin{aligned} \overset{s}{\mathcal{L}}_{\xi} \Psi &= \frac{d}{dt} \overset{s}{\Psi}_t(x)|_{t=0} \\ &= \mathfrak{d}_{\xi} \Psi - \frac{1}{4} d\xi \Psi + \frac{1}{4} \xi^{\kappa} \omega_{\alpha\kappa\beta} \underline{\gamma}^{\alpha} \underline{\gamma}^{\beta} \end{aligned} \quad (88)$$

with the $\underline{\gamma}^{\alpha}$'s being matrix representations of the γ^{α} 's. Of course,

$$\overset{s}{\mathcal{L}}_{\xi} \Psi = D_{\xi} \Psi - \frac{1}{8} (D_{\alpha} \xi_{\beta} - D_{\beta} \xi_{\alpha}) \underline{\gamma}^{\alpha} \underline{\gamma}^{\beta} \Psi, \quad (89)$$

6 The Spinor Lie derivative of the Metric Field

We want to extend the spinor Lie derivative that we just defined for Clifford and spinor fields to sections of the tensor bundle. The general case will not be discussed today. Here we give the

Definition 27 The spinor Lie derivative of the metric field $\mathbf{g} = \eta_{\alpha\beta} \gamma^{\alpha} \otimes \gamma^{\beta}$ in the direction of the arbitrary vector field ξ is

$$\overset{s}{\mathcal{L}}_{\xi} \mathbf{g}(x) = \lim_{t \rightarrow 0} \frac{\check{\mathbf{g}}_t(x) - \mathbf{g}(x)}{t}, \quad \check{\mathbf{g}}_t(x) := \eta_{\alpha\beta} \check{\gamma}_t^{\alpha}(x) \otimes \check{\gamma}_t^{\beta}(x) \quad (90)$$

Proposition 28 For any vector field ξ , it is $\overset{s}{\mathcal{L}}_{\xi} \mathbf{g} = 0$.

Proof. Indeed we have

$$\begin{aligned} \overset{s}{\mathcal{L}}_{\xi} \mathbf{g}(x) &= \lim_{t \rightarrow 0} \frac{\eta_{\alpha\beta} \check{\gamma}_t^{\alpha}(x) \otimes \check{\gamma}_t^{\beta}(x) - \eta_{\alpha\beta} \gamma_t^{\alpha}(x) \otimes \gamma_t^{\beta}(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\eta_{\alpha\beta} (u_t^{-1}(x) \gamma^{\alpha}(x) u_t(x)) \otimes (u_t^{-1}(x) \gamma^{\beta}(x) u_t(x) - \eta_{\alpha\beta} \gamma_t^{\alpha}(x) \otimes \gamma_t^{\beta}(x))}{t} \\ &= \lim_{t \rightarrow 0} \frac{\eta_{\alpha\beta} \Lambda_{t\kappa}^{\alpha}(x) \gamma^{\kappa}(x) \otimes \Lambda_{t\iota}^{\beta}(x) \gamma^{\iota}(x) - \eta_{\alpha\beta} \gamma_t^{\alpha}(x) \otimes \gamma_t^{\beta}(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\mathbf{g}(x) - \mathbf{g}(x)}{t} = 0. \end{aligned} \quad (91)$$

■

Remark 29 So, technically speaking $\overset{s}{\mathcal{L}}_{\xi}$ is a perfectly well defined operator acting in the Clifford and spin-Clifford bundles having the nice properties exhibited above. It defines a derivation in the Clifford bundle since it does pass to the quotient $\tau M/I = \mathcal{Cl}(M, \mathbf{g})$, where I is the bilateral ideal generated by elements of the form $(a \otimes b + b \otimes a - 2\mathbf{g}(a, b))$ with $a, b \in \sec \bigwedge^1 T^*M$.

¹⁴The column spinor fields used in physical textbooks.

Of course, when ξ is a Killing vector field we have $\mathcal{L}_\xi \mathcal{C} = \overset{s}{\mathcal{L}}_\xi \mathcal{C}$

Remark 30 *A definition of Lie derivative of tensor and spinor fields such that it always annihilates \mathbf{g} has be given by Bourguignon & Gauduchon [3] using a very different method.*

7 Other Definitions of Lie Derivatives of Spinor Fields.

Once again we recall that the definition of the *spinor* Lie derivative of spinor fields generated by Killing vector fields has first given by Lichnerowicz [22] and taken valid (as a definition) for arbitrary diffeomorphisms generated by arbitrary vector fields by Kosmann [18, 19, 20]. A “justification” of Kosmann’s formula for the case where ξ is a Killing vector field is given, e.g., in [1]. There, it is imposed that the Lie derivative be a *derivation* on the Clifford bundle, by passing to the quotient the action of the Lie derivative on $\tau M/I = \mathcal{Cl}(M, \mathbf{g})$ which necessarily implies that ξ must be a Killing vector field. The Lie derivative is then obtained using an analogy with the concept of covariant derivative in the following way. First, one recall that the covariant derivative of a Clifford field $\mathcal{C} \in \sec \mathcal{Cl}(M, \mathbf{g})$ can be written as

$$D_\xi \mathcal{C} = \mathfrak{d}_\xi \mathcal{C} + \frac{1}{2}[\omega_\xi, \mathcal{C}] \quad (92)$$

and the covariant derivative of a representative in the Clifford bundle of a DHSF can be written as

$$\overset{s}{D}_\xi \psi_{\Xi_0} = \mathfrak{d}_\xi \psi_{\Xi_0} + \frac{1}{2}\omega_\xi \psi_{\Xi_0}. \quad (93)$$

Next, showing that for a Killing vector field ξ and $X \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{Cl}(M, \mathbf{g})$ the standard Lie derivative of X is

$$\mathcal{L}_\xi X = \mathfrak{d}_\xi X + \frac{1}{4}[L(\xi) + d\xi, X] \quad (94)$$

$$= D_\xi X + \frac{1}{4}[d\xi, X] \quad (95)$$

it is *postulated* that.

$$\mathcal{L}_\xi \psi_{\Xi_0} := \mathfrak{d}_\xi \psi_{\Xi_0} + \frac{1}{4}(L(\xi) + d\xi)\psi_{\Xi_0} \quad (96)$$

$$= \overset{s}{D}_\xi \psi_{\Xi_0} + \frac{1}{4}d\xi \psi_{\Xi_0} \quad (97)$$

which is the Kosmann’s formula.

Remark 31 *We emphasize that we get the above formulas with a very different procedure¹⁵, namely by finding a geometrical motivated definition for the image*

¹⁵Without introducing at starting the use of Levi-Civita connections, something that seems unjustifiable if we want a meaningful notion of Lie derivative.

of Clifford and spinor fields generated by one-parameter groups of diffeomorphisms associated to an arbitrary smooth vector field ξ .

Remark 32 We also mention that in [14] a Lie derivative of a spinor field is also defined using analogy with the covariant derivative and a formula is obtained similar to the formulas that we get for $\overset{s}{\mathcal{L}}$ but with an extra term, which authors claim to be necessary in order to have agreement with the Lie derivative of general tensor fields with the ones obtained from the Lie derivative of general tensors fields represented by a tensor product of spinor fields. As already mentioned above this important issue will be discussed in another publication. Anyway, we emphasize that our definition of $\overset{s}{\mathcal{L}}$ seems perfectly consistent with the Clifford and spin-Clifford bundles formalism.

Remark 33 It is worth to mention that an equation similar Eq.(82) has also been obtained in [9] using the general concept of Lie differentiation in the elegant theory of gauge natural bundles. The theory of Lie differentiation in gauge natural bundles was originally introduced by [7] and developed by Klokár and collaborators¹⁶. It is reviewed with emphasis in physical applications in [8, 11, 12]. Authors [11, 12] claim that [9] succeeded in given a geometrical meaning for the Kosmann definition, but the case is that what has been done there was to postulated a particular lifting of the vector field $\xi \in \sec TM$ to the tangent bundle to $P_{\text{Spin}_{1,3}^c}(M, \mathbf{g})$ such that the definition of (generalized) Lie derivative of a spinor field results in Kosmann's formula. This is a sophisticated way to get the same result we get using a very simple and intuitive path.

Remark 34 To obtain the Euler-Lagrange equations from the principle of stationary action for a system consisting of a spinor field, the electromagnetic and gravitational field¹⁷ implies in giving a clear definition of what we mean by the variation of these fields. If the variations of the Clifford fields representing the electromagnetic field and of the spinor fields representing matter if given by $\overset{s}{\mathcal{L}}\xi$ we will have as a consequence that the metric \mathbf{g} defined by the cotetrad fields will have null variation when the cotetrad fields are varied.

Remark 35 We recall here that when we represent the gravitational field by the cotetrad fields γ^α in a Riemann-Cartan theory (see [29]) we need, in order to obtain covariant conservation laws for the matter and electromagnetic fields to make use of vertical ($\delta_{\mathbf{v}}\gamma^\alpha = \Lambda_\beta^\alpha \gamma^\beta$, with Λ_β^α a local Lorentz rotation) and horizontal ($\delta_{\mathbf{h}}\gamma^\alpha = -\mathcal{L}_\times \gamma^\alpha$) variations. However, existence of genuine (not the covariant ones) conservation laws requires the existence of appropriated Killing vector fields and consistency of the formalism requires in that case that

$$\delta_{\mathbf{v}}\gamma^\alpha = -\Lambda_\beta^\alpha \gamma^\beta = -\mathcal{L}_\times \gamma^\alpha. \quad (98)$$

These constrained variations of the γ^α have been used in the theory of the gravitational field developed in [29] and with improvements in [30]. Now, taking into

¹⁶See [15].

¹⁷With the gravitational field represented by the cotetrad fields $\{\gamma^\alpha\}$.

account that it is $\overset{s}{\mathcal{L}}_{\xi}g = 0$, it is the case that $\overset{s}{\mathcal{L}}_{\xi}\gamma^{\alpha} = \Lambda_{\beta}^{\alpha}\gamma^{\beta}$ we see that a consistent Lagrangian formalism for fields represented by Clifford fields (this including the representatives in the Clifford bundle of Dirac-Hestenes spinor fields) may be based on taken variations of the fields ϕ entering the Lagrangian density as being $\overset{s}{\mathcal{L}}_{\xi}\phi$. This will be discussed in another publication.

8 Conclusions

In this paper we claim to have given a geometrical motivated definition for a Lie derivative of spinor fields in a Lorentzian structure (M, g) by finding an appropriated image for Clifford and spinor fields under a diffeomorphism generated by an arbitrary vector field ξ . We called such operator the *spinor Lie derivative*, denoted $\overset{s}{\mathcal{L}}_{\xi}$ which is such that $\overset{s}{\mathcal{L}}_{\xi}g = 0$ for arbitrary vector field ξ . We compared our definitions and results with the many others appearing in literature on the subject.

References

- [1] Benn I. M., and Tucker, R. W., *An Introduction to Spinors and Geometry with Applications in Physics*, Adam Hilger, Bristol and New York, 1987
- [2] Blaine Lawson, H., Jr., Michelsohn, M.-L., *Spin Geometry*, Princeton University Press, New Jersey.(1989).
- [3] Bourguignon, J.-P. and Gauduchon, P., Spineurs, Operateurs de Dirac et Variations de Métriques, *Comm. Math. Phys.* **144**, 581-599 (1992)
- [4] Choquet-Bruhat, Y., DeWitt- Morette, C., and Deillard-Bleick, M., *Analysis, Manifolds and Physics* (revised edition), North-Holland, Amsterdam , 1982.
- [5] Crummeyrole, A., *Orthogonal and Symplectic Clifford Algebras*, Kluwer Acad. Publ., Dordrecht, 1.990.
- [6] Dabrowski, L. and Percacci, R., Spinors and Diffeomorphisms, *Commun. Math. Phys.* **106**, 691-704 (1986).
- [7] Eck, D. J, Gauge Natural Bundles and Gauge Theories, *Mem. Amer. Math. Soc.* **33**, number 247, (1981).
- [8] Fatibene, L. and Francaviglia, M, *Natural and Gauge Natural Formalism for Classical Field Theories*, Kluwer Academic Publ., Dordrecht, 2003.
- [9] Fatibene, L., Ferraris, Francaviglia, M, and Godina, M., A Geometric Definition of Lie Derivative for Spinor Fields, in *Proc. of the 6th International Conference on Differential Geometry and Applications*, August 28th-September 1st 1995 (Brno, Czech Republic), Masaryk University, Brno. [arXiv:gr-qc/960800]

- [10] Geroch, R., Spinor Structure of Space-Times in General Relativity I, *J. Math. Phys.* **9**, 1739-1744 (1968).
- [11] Godina, M., and Matteucci, P., The Lie Derivative of Spinor Fields: Theory and Applications, *Int. J. Geom. Methods Mod. Phys.* **2**, 159-188 (2005) [arXiv:math/0504366 [math.DG]]
- [12] Godina, M., and Matteucci, P., *Reductive G-Structures and Lie Derivatives* [arXiv:math/0201235v2 [math-DG]]
- [13] Gürsey, F., Introduction to Group Theory, in De Witt, C. and De Witt, B. (eds.), *Relativity, Groups and Topology*, Gordon and Breach, New York, 1963.
- [14] Hurley, D. J. and Vandyck, M. A., *Geometry, Spinors and Applications*, Springer (published in association with Praxis Publ., Chichester), 2000.
- [15] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993
- [16] Kobayashi, S., and Nomizu, K., *Foundations of Differential Geometry*, vol I, J. Wiley-Interscience, New York, 1981.
- [17] Kosmann, Y. Dérivées de Lie des Spineurs, *C. R. Acad. Sci. Paris, Série A* **262**, 289-292, (1966).
- [18] Kosmann, Y., Dérivées de Lie des Spineurs. Applications, *C. R. Acad. Sci. Paris , Paris, Série A* **262**, 394-397 (1966)
- [19] Kosmann, Y., Propriétés des Dérivations de l'algèbre des Tenseurs-Spineurs, *C. R. Acad. Sc. Paris, Série A* **264**, 355-358. (1967)
- [20] Kosmann, Y., Dérivées de Lie des Spineurs (thèse), *Ann. Mat. Pura Appl. IV*, 317-395, (1972).
- [21] Lawson, H. B. Jr. and Michelson, M-L., *Spin Geometry*, Princeton University Press, Princeton, 1989.
- [22] Lichnerowicz, A., Spineurs Harmoniques, *C. R. Acad. Sci. Paris* **257**, 7-9 (1963).
- [23] Lounesto, P., *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge, 1997.
- [24] Lovelock, D., and Rund, H., Tensors, *Differential Forms, and Variational Principles*, J. Wiley & Sons, New York, 1975.
- [25] Mosna, R. A. and Rodrigues, W. A. Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, *J. Math. Phys.* **45**, 2945-2966 (2004). [arXiv:math-ph/0212033]
- [26] Penrose, R. and Rindler, W., *Spinors and Spacetime*, vol.2, Spinor and Twistor Methods in Spacetime Geometry, Cambridge Univ. Press, Cambridge, 1986.

- [27] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, *J. Math. Phys.* **45**, 2908-2994 (2004). [arXiv:math-ph/0212030]
- [28] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, *J. Math. Phys.* **45**, 2908-2994 (2004) [arXiv:math-ph/0212030].
- [29] Rodrigues, W. A. Jr. and Capelas de Oliveira, E., *The Many Faces of Maxwell, Dirac and Einstein Equation*, A Clifford Bundle Approach, Lecture Notes in Physics 722, Springer, Heidelberg, 2007. A preliminary enlarged second edition may be found at <http://www.ime.unicamp.br/~walrod/mde100214.pdf>
- [30] Rodrigues, W. A. Jr., Nature of the Gravitational Field and its Legitimate Energy-Momentum Tensor, *Rep. Math. Phys.* **69**, 265-279 (2012) [arXiv:1109.5272 [math-ph]]
- [31] Weinberg, S., *Gravitation and Cosmology; Principles and Applications of the General Theory of Relativity*, J. Wiley and Sons, New York, 1972